TOPIC — 2nd order Differential Equation with variable co-efficients.

Method of variation of parameters

B.Sc III (Math Hons)

Date —

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Study Material → This Method helps us in finding the general solution of the Equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R. \quad \text{—— (1)}$$

when the complementary function is known.

Let the C.F be

$$y = au + bv$$

where a and b are constant and u and v are functions of x.

Then u and v must be solution of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \quad \text{—— (2)}$$

Hence $u_2 + Pu_1 + Qu = 0$

and $v_2 + Pv_1 + Qv = 0$ } —— (3)

When $R \neq 0$

$au + bv$ will not represent the complete solution of (1)

we consider

$$y = Au + Bv \quad \text{—— (4)}$$

is complete solution of (1)

where A and B are no longer constants but function of x to be so choosen that (1) shall be satisfied.

Thus the form of y is the same for the two equations (2) and (1),

but the constants which occur in the former case are changed in the latter into functions of the independent variable.

This is why this Method is known as variation of parameters.

We have now two unknown quantities A and B, in Terms of y has been expressed by (4)

For A and B

we requires two equations containing them.

In order to get such equations we must impose two conditions on them. We have impose one condition

y is complete solution of (1)

for our convenience,

let us take

$$uA_1 + vB_1 = 0 \quad \text{---} \quad (5)$$

Now Differentiating (4) and using (5) we get

$$y_1 = Au_1 + Bv_1 \quad \text{---} \quad (6)$$

$$y_2 = Au_2 + A_1u_1 + Bv_2 + B_1v_1 \quad \text{---} \quad (7)$$

putting the values of y, y_1, y_2 obtained by (4) (6) and (7) in (1) we get

$$Au_2 + A_1u_1 + Bv_2 + B_1v_1 + P(Au_1 + Bv_1) + Q(Au + Bv) = R$$

or $A(u_2 + P u_1 + Q u) + B(v_2 + P v_1 + Q v) + A_1 u_1 + B_1 v_1 = R$ — (7)

using (3)

$$A \cdot 0 + B \cdot 0 + A_1 u_1 + B_1 v_1 = R$$

or $A_1 u_1 + B_1 v_1 = R$ ——— (8)

Also $A_1 u + B_1 v = 0$

On solving

$$\left.\begin{aligned} A_1 &= \frac{dA}{dx} = \frac{-vR}{uv_1 - u_1 v} = \frac{-vR}{W} \\ \text{and } B_1 &= \frac{dB}{dx} = \frac{uR}{uv_1 - u_1 v} = \frac{uR}{W} \end{aligned}\right\} \text{ — (9)}$$

where W = Wronskian of u and v

$$= \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = uv_1 - u_1 v$$

Integrating (9)

$$\left.\begin{aligned} A &= f(x) + c_1 \\ B &= g(x) + c_2 \end{aligned}\right\} \text{ say}$$

Putting these values in (4), the required General Solution of (1) is

$$y = c_1 u + c_2 v + u f(x) + v g(x)$$

c_1 and c_2 being arbitrary constants.

So the Method of variation of parameters is used when

(i) we are asked to solve the equation by variation of parameters.

(ii) C.F is known easily but the particular integral of the same differential Equation can not be obtained by any previous Methods.